ISRAEL JOURNAL OF MATHEMATICS **94** (1996), 21–27

ON NONIMMERSIBILITY OF COMPACT HYPERSURFACES INTO A BALL OF A SIMPLY CONNECTED SPACE FORM*

BY

FERNANDO GIMÉNEZ

Departamento de Matemática Aplicada
E.T.S.I. Industriales, Universidad Politécnica de Valencia, Valencia, Spain

AND

VICENTE MIQUEL

Departamento de Geometría y Topología
Universidad de Valencia, Burjasot, Valencia, Spain
e-mail: miquel@mac.uv.es

ABSTRACT

We give a nonimmersibility theorem of a compact manifold with nonnegative scalar curvature bounded from above into a geodesic ball of a simply connected space form.

1. Introduction

There are many theorems about nonimmersibility of compact riemannian manifolds with sectional curvature bounded from above inside a metric ball of a simply connected space form (see [Ja], [Be1 and 2], [JK], [Ko] and [CI]). Recently, S. Deshmukh and M. A. Al-Gwaiz ([DA]) have proved the following theorem, which works with Ricci curvatures instead of sectional curvatures,

THEOREM A ([DA]): *Let $\mathbb{C}^n$ be the Euclidean space of dimension $2n$ and M be a compact $(2n-1)$-dimensional Riemannian manifold whose Ricci curvature ρ and scalar curvature τ satisfy $\rho(X,X)+\tau \geq 0$ and $\rho(X,X) < 2(n-1)R^{-2}$ for*

* Work partially supported by a DGICYT Grant No. PB91-0324 and by the E.C. Contract CHRX-CT92-0050 "G.A.D.G.E.T. II"
Received September 28, 1994

each unit vector field X on M and some positive constant R. Then no isometric immersion of M into $\mathbb{C}^n$ is contained in a ball of radius R.

In this note we give a generalization of this theorem in the following sense: we remove the restriction of even dimension of the ambient manifold and allow it to be any simply connected space form. Moreover, we remark that the upper bound on the Ricci curvature can be weakened to an inequality between the total scalar curvature and the volume of M.

More precisely, given a real number λ, let us consider the function

$$s_\lambda(t) = \begin{cases} \dfrac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} & \text{if } \lambda > 0 \\ t \ \text{ if } \lambda = 0 \\ \dfrac{\sinh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}} & \text{if } \lambda < 0, \end{cases}$$

and let $\mathbb{K}^n(\lambda)$ be the simply connected space form of dimension n and sectional curvature λ. Then we prove the following theorem:

THEOREM 1: *Let M be a compact $(n-1)$-dimensional Riemannian manifold whose Ricci curvature ρ and scalar curvature τ satisfy*

$$\rho(X,X) + \tau \geq (n+1)(n-2)\lambda \quad (\text{and } \tau \geq 0 \text{ if } \lambda < 0)$$

for each unit vector field X on M and

$$\int_M \tau dM \leq (n-1)(n-2)\operatorname{vol}(M)s_\lambda^{-2}(R)$$

for some positive constant R ($R \leq (1/\sqrt{\lambda}) \arcsin\sqrt{n/(n+1)}$ if $\lambda > 0$). If there is an isometric immersion ψ of M into $\mathbb{K}^n(\lambda)$ contained in a geodesic ball of radius R, then $\psi(M)$ is the boundary of this geodesic ball.

For R as in Theorem 1, let $B_R(o)$ be a geodesic ball of radius R and centre o in $\mathbb{K}^n(\lambda)$. Let $r\colon \mathbb{K}^n(\lambda) \longrightarrow \mathbb{R}$ be the distance to o in $\mathbb{K}^n(\lambda)$, and denote also by r the composition $r \circ \psi$, ψ being an isometric immersion of M in $\mathbb{K}^n(\lambda)$. Let N be a unit normal vector field to M. Let us denote by ∂_r the gradient of r in $\mathbb{K}^n(\lambda)$, and by $\partial_r^\top$ the vector field on M defined by

$$\partial_r^\top(m) = \psi_*^{-1}(\partial_r(\psi(m)) - \langle \partial_r(\psi(m)), N(m)\rangle N(m)) \quad \text{for every } m \in M.$$

Let us define the tangent vector field u on M by $u(m) = s_\lambda(r(m))\partial_r^\top(m)$. Even if N is defined only up to a sign, u is globally well defined.

Theorem 1 is an obvious consequence of the following one:

THEOREM 2: *Let M be a compact $(n-1)$-dimensional Riemannian manifold whose Ricci curvature ρ and scalar curvature τ satisfy*

$$\int_M \{\rho(u,u) + \tau|u|^2\} dM \geq (n+1)(n-2)\lambda \int_M |u|^2 dM \tag{2.1}$$

(and $\tau \geq 0$ if $\lambda < 0$) and

$$\int_M \tau dM \leq (n-1)(n-2)\operatorname{vol}(M) s_\lambda^{-2}(R). \tag{2.2}$$

If there is an isometric immersion ψ of M into $\mathbb{K}^n(\lambda)$ contained in $B_R(o)$, then $\psi(M)$ is the boundary of this geodesic ball.

Theorem 2 will be proved in section 3. In section 2 we shall prove a new Minkowski formula for hypersurfaces of $\mathbb{K}^n(\lambda)$. This is one of the main ingredients in the proof of Theorem 2. Another one is the choice of the vector field u, different from that used in [DA], which allows us to drop the condition of even dimension. The third ingredient is the integral formula (10), taken from [Ya] and used before by S. Deshmukh and M. A. Al-Gwaiz in [DA]) in their proof of Theorem A.

2. General Minkowski formulae

Let $\psi: M \longrightarrow \overline{M}$ be an isometric immersion of a compact Riemannian manifold M, of dimension $n-1$, into an n-dimensional Riemannian manifold $\overline{M}$. Let $o \in \overline{M}$ and let r, ∂_r and $\partial_r^\top$ be defined on M and on $\overline{M}$ as in section 1, where the role of $\mathbb{K}^n(\lambda)$ is played by $\overline{M}$. Let $\overline{\nabla}$ and $\overline{\Delta}$ denote the covariant derivative and the Laplacian respectively on $\overline{M}$. Let us denote by ∇ and Δ the corresponding operators on M. Let N be a unit normal vector on M and let L be the associated Weingarten map of M in $\overline{M}$. In the local computations below, we shall denote by the same letter a local vector field X on M and its image $\psi_* X$.

If we denote by $S(r)$ the $(1,1)$-tensor on $\overline{M}$ defined by

$$S(r)(A) = -\overline{\nabla}_A \partial_r \quad \text{for every } A \text{ tangent to } \overline{M},$$

then the following formula is well known (cf. [GW]):

$$\overline{\nabla}^2 r(A,B) = -\langle S(r)A, B\rangle \quad \text{for every } A, B \text{ tangent to } \overline{M}, \tag{1}$$

which implies

$$\overline{\Delta} r = \operatorname{tr} S(r). \tag{2}$$

Let us observe that $S(r)\partial_r = 0$ and $S(r)$ restricted to the vectors tangent to the geodesic sphere $\partial B_r(o)$ of $\overline{M}$ of centre o and radius r is the Weingarten map of this sphere, and $\operatorname{tr} S(r)$ is $(n-1)$ times the mean curvature of this sphere.

On the other hand, an easy computation shows that

$$\nabla^2 r(X,Y) = \overline{\nabla}^2 r(X,Y) + \langle LX, Y\rangle\langle \partial_r, N\rangle \tag{3}$$

for every X, Y tangent to M, where L is the Weingarten map of the immersion $\psi\colon M \longrightarrow \overline{M}$.

If $\{e_i\}_{i=1}^{n-1}$ is a local orthonormal frame of vector fields tangent to M and H denotes the mean curvature of M, from (2) and (3), it follows that

$$\Delta r = \sum_{i=1}^{n-1} \langle S(r)(e_i - \langle e_i, \partial_r\rangle\partial_r), e_i\rangle - (n-1)H\langle N, \partial_r\rangle. \tag{4}$$

If $f(r)$ is any C^2 function, it follows from (4) that

$$\Delta f(r) = -f''(r)|\partial_r^\top|^2 + f'\{\sum_{i=1}^{n-1}\langle S(r)(e_i - \langle e_i,\partial_r\rangle\partial_r), e_i\rangle - (n-1)H\langle N, \partial_r\rangle\}. \tag{5}$$

Integration of this formula over M gives what we could call a general Minkowski formula. Since we are interested in a particular one when $\overline{M} = \mathbb{K}^n(\lambda)$ we must select an appropriate function $f(r)$. Before doing it we introduce some new functions

$$c_\lambda(t) = \begin{cases} \cos(\sqrt{\lambda}t) & \text{if } \lambda > 0 \\ 1 & \text{if } \lambda = 0 \\ \cosh(\sqrt{|\lambda|}t) & \text{if } \lambda < 0, \end{cases}$$

$$co_\lambda(t) = \frac{s'_\lambda(t)}{s_\lambda(t)} = \frac{c_\lambda(t)}{s_\lambda(t)} = \begin{cases} \sqrt{\lambda}\cot(\sqrt{\lambda}t) & \text{if } \lambda > 0 \\ 1/t & \text{if } \lambda = 0 \\ \sqrt{|\lambda|}\coth(\sqrt{|\lambda|}t) & \text{if } \lambda < 0. \end{cases}$$

The functions s_λ and c_λ satisfy the following computation rules:

$$s'_\lambda = c_\lambda, \quad c'_\lambda = -\lambda s_\lambda, \quad c_\lambda^2 + \lambda s_\lambda^2 = 1, \quad s_{4\lambda} = s_\lambda c_\lambda, \quad c_{4\lambda} = c_\lambda^2 - \lambda s_\lambda^2.$$

In $\mathbb{K}^n(\lambda)$, the operator $S(r)$ restricted to the tangent space of a geodesic sphere $\partial B_r(o)$ of centre o and radius r is given by (cf. [Gr])

$$S(r) = -co_\lambda(r)\,\mathbf{Id}. \tag{6}$$

Since $e_i - \langle e_i, \partial_r\rangle\partial_r$ is tangent to $\partial B_r(o)$, from (5) and (6) one gets

$$\Delta f(r) = (-f''(r) + co_\lambda(r)f'(r))|\partial_r^\top|^2 - (n-1)(co_\lambda(r)f' + H\langle N, f'\partial_r\rangle). \tag{7}$$

Then, if we take $f' = s_{4\lambda}$ (that is $f = -c_{4\lambda}/(4\lambda)$ if $\lambda \neq 0$ and $f(r) = (1/2)r^2$ if $\lambda = 0$), we have

$$\Delta f(r) = \lambda s_\lambda^2(r)|\partial_r^\top|^2 - (n-1)c_\lambda^2(r) - (n-1)Hc_\lambda(r)\langle N, s_\lambda(r)\partial_r\rangle. \tag{8}$$

Now we denote $\alpha(m) = \langle N(m), s_\lambda(r(m))\partial_r(m)\rangle$ for every $m \in M$ and u as in section 1. Although N (and then α and H) is (globally) defined only up to a sign, the product αH is globally well defined. Then, we can integrate (8) over M and apply Stokes Theorem to get

$$0 = \int_M \{\lambda|u|^2 - (n-1)c_\lambda(c_\lambda + \alpha H)\}dM. \tag{9}$$

This special Minkowski formula differs from the standard ones (cf. [Hs] and [MR]) in the first summand and in the factor c_λ. It is just this factor c_λ which we need to combine formula (9) with formula (10) below, and this is the reason why we use the function $f = -c_{4\lambda}/(4\lambda)$.

3. Proof of Theorem 2

First, let us suppose that $u(m) = 0$ for every $m \in M$. Then $\partial_r^\top = 0$ and $dr(X) = \langle\partial_r, X\rangle = 0$ for every X tangent to M. Then r is constant on M and M is a geodesic sphere of radius r in $\mathbb{K}^n(\lambda)$. From the Gauss equation for a submanifold, one gets that, in this case, $\tau = (n-1)(n-2)(1/(s_\lambda^2(r))$. Since s_λ is an increasing function and $r \leq R$ by hypothesis, we have

$$\int_M \tau dM \geq (n-1)(n-2)\,\mathrm{vol}(M)s_\lambda^{-2}(R).$$

This inequality and the hypothesis (2.2) imply the equality in (2.2). Then $r = R$ and this proves the theorem.

Let us now suppose that $u \neq 0$ on an open set of M.

The following integral formula is well known (see [Ya, page 41]):

$$\int_M \{\rho(X,X) + \tfrac{1}{2}|\mathcal{L}_X g|^2 - |\nabla X|^2 - |\delta X^\flat|^2\} dM = 0 \tag{10}$$

for every vector field X tangent to M, where g is the metric of M, $\mathcal{L}_X$ denotes the Lie derivative respect to X, $X^\flat$ is the 1-form on M defined by $X^\flat(Y) = \langle X, Y\rangle$ and δ denotes the coderivative on M induced by $g \equiv \langle , \rangle$.

Now we are going to compute the terms in the integrand of (10) for $X = u$. First we compute $\nabla_{e_i} u$,

$$\nabla_{e_i} u = \nabla_{e_i}(s_\lambda \partial_r^\top) = c_\lambda \langle e_i, \partial_r\rangle \partial_r^\top + s_\lambda \nabla_{e_i} \partial_r^\top,$$

but, denoting by $A^\top$ the component tangent to M of an arbitrary vector A and using (6), we have

$$\begin{aligned}\nabla_{e_i}\partial_r^\top &= (\overline{\nabla}_{e_i}(\partial_r - \langle \partial_r, N\rangle N))^\top \\ &= -(S(r)(e_i - \langle e_i, \partial_r\rangle \partial_r))^\top + \langle \partial_r, N\rangle L e_i \\ &= co_\lambda e_i - co_\lambda \langle e_i, \partial_r\rangle \partial_r^\top + \langle \partial_r, N\rangle L e_i,\end{aligned}$$

then

$$\nabla_{e_i} u = c_\lambda\, e_i + \alpha\, L e_i.$$

From this expression, straightforward computations give

$$\begin{aligned}|\nabla u|^2 &= (n-1)c_\lambda^2 + 2\,\alpha\,(n-1)\,c_\lambda H + \alpha^2 |L|^2, \\ |\delta u^\flat|^2 &= c_\lambda^2\,(n-1)^2 + \alpha^2\,(n-1)^2\,H^2 + 2\,(n-1)^2\,c_\lambda\,\alpha\,H, \\ |\mathcal{L}_u g|^2 &= 4\,(n-1)\,c_\lambda^2 + 8\,(n-1)\,c_\lambda\,\alpha\,H + 4\,\alpha^2\,|L|^2,\end{aligned}$$

and, using Gauss formula (which gives $\tau = (n-1)(n-2)\lambda + (n-1)^2 H^2 - |L|^2$) and the fact that, from the definitions of α and u, $\alpha^2 = s_\lambda^2 - |u|^2$, we get

$$\begin{aligned}&\rho(u,u) + \frac{1}{2}|\mathcal{L}_u g|^2 - |\nabla u|^2 - |\delta u^\flat|^2 \\ &= \rho(u,u) + (\tau - (n-1)(n-2)\lambda)|u|^2 - \tau s_\lambda^2 + (n-1)(n-2)\lambda s_\lambda^2 \\ &\quad - (n-1)(n-2)c_\lambda^2 - 2(n-1)(n-2)c_\lambda \alpha H.\end{aligned} \tag{11}$$

Now, from (9), (10) and (11), we get

$$\int_M \{\rho(u,u) + (\tau - (n+1)(n-2)\lambda)|u|^2 - s_\lambda^2 \tau + (n-1)(n-2)\} dM = 0.$$

This equality, together with the hypothesis (2.1) and (2.2) of Theorem 2, the fact that $s_\lambda(r)$ is an increasing positive function (and the hypothesis $\tau \geq 0$ if $\lambda < 0$) imply $r = R$ and we are as in the case $u = 0$ everywhere.

References

[Be1] M. A. Beltagy, *Isometric embedding of a compact Riemannian manifold into a sphere,* Indian Journal of Pure and Applied Mathematics **14** (1983), 1406–1411.

[Be2] M. A. Beltagy, *Isometric embedding of a compact Riemannian manifold into hyperbolic space,* Journal of the Indian Mathematical Society (N.S.) **48** (1984), 81–87.

[CI] L. Coghlan and Y. Itokawa, *On the sectional curvature of compact hypersurfaces,* Proceedings of the American Mathematical Society **109** (1990), 215–221.

[DA] S. Deshmukh and M. A. Al-Gwaiz, *Compact hypersurfaces in even dimensional euclidean space and in the sphere,* The Quarterly Journal of Mathematics. Oxford. **45** (1994), 151–157.

[Gr] A. Gray, *Tubes,* Addison-Wesley, Reading, 1990.

[GW] R. E. Greene and H. Wu, *Function Theory on Manifolds Which Possess a Pole,* Lecture Notes in Mathematics, Vol. 699, Springer-Verlag, Berlin, 1979.

[Hs] C. C. Hsiung, *Some integral formulas for closed hypersurfaces,* Mathematica Scandinavica **2** (1954), 286–294.

[Ja] H. Jacobowitz, *Isometric embedding of a compact Riemannian manifold into euclidean space,* Proceedings of the American Mathematical Society **40** (1973), 245–246.

[JK] L. Jorge and D. Koutroufiotis, *An estimate for the curvature of bounded submanifolds,* American Journal of Mathematics **103** (1981), 711–725.

[Ko] D. Koutroufiotis, *Elementary geometric applications of a maximum principle for nonlinear elliptic operators,* Archiv der Mathematik **24** (1973), 97–99.

[MR] S. Montiel and A. Ros, *Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures,* in *Differential Geometry* (B. L. Lawson and K. Tenenblat, eds.), Longman Sci. and Tech., Essex, 1991, pp. 279–296.

[Ya] K. Yano, *Integral Formulas in Riemannian Geometry,* Marcel Dekker, New York, 1970.